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The lattice equations of the Toda type with an interaction between a few neighbourhoods

N V Ustinov

Theoretical Physics Department, Kaliningrad State University, Al Nevsky street 14, 236041, Kaliningrad, Russia

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Abstract

The sets of integrable lattice equations, which generalize the Toda lattice, are considered. The hierarchies of the first integrals and infinitesimal symmetries are found. The properties of the multi-soliton solutions are discussed.

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1. Introduction

The first differential-difference system studied by the methods of the theory of solitons was the famous Toda lattice [1]

$$\ddot{\sigma}_k = \mathrm{e}^{\sigma_{k+1}-\sigma_k} - \mathrm{e}^{\sigma_k-\sigma_{k-1}}. \tag{1}$$

The multi-soliton solutions of this system were obtained in [2]. That the Toda lattice belongs to a class of completely integrable systems was established in [3, 4] in periodic and infinite dimensional cases.

The development of the theory of solitons led to various generalizations of the Toda lattice (see, e.g., [5]). Although this system was originally suggested as one describing the behaviour of particles interacting with the nearest neighbourhoods, it also appears as a limit in the elliptic Calogero–Moser system [6], where all the particles interact with each other. The problem of the classification of integrable lattices of the Toda type with an interaction between two nearest neighbourhoods was considered in [7, 8]. Recently, the new classes of the integrable lattice equations that include the Toda lattice as a particular case [9–11] and the nonlocal two-dimensional generalization of equations (1) [12] were investigated.

Significant attention has been paid during the past ten years to revealing the role of the integrable multi-particle systems in nonlinear sciences. It was shown that equations (1) and other well-known lattice equations are the discrete symmetries (Darboux–Bäcklund transformations) of the Kadomtsev–Petviashvilii hierarchies [13, 14]. Also, these systems play an important role in nonperturbative string theory and D-brane theory [15]. In particular, the Toda lattice is connected with the Nahm equations [16, 17] and determines the behaviour of the collective coordinates of the branes [18, 19] and of the massless supersymmetric gauge theories in the low-energy sector [20–22].

In the present paper we consider the generalization of equations (1) in the cases of systems of particles whose motion is immediately affected by the finite number of nearest neighbourhoods. These lattices are the members of the one-parameter subvarieties of two different sets of the two-parameter two-field integrable lattice equations studied in [10, 11]. It is remarkable that, as for the Toda lattice case, there exists a connection of the lattices considered with the Nahm equations and their continuous limit is the Boussinesque equation. The reductions of the lattice equations and their direct and dual Lax pairs are presented in section 2. In this section, we also give the expressions for the first integrals of the lattices and for the asymptotic expansions of the solutions of the Lax pairs. The Darboux transformation technique [23] is applied in section 3 to obtain the hierarchies of the infinitesimal symmetries and the multi-soliton solutions of the lattice equations.

2. Lattice equations of the Toda type

Let us consider two infinite sets ($l \in \mathbb{N}$) of lattice equations

$$\ddot{\sigma}_k = C\left(\exp\left(\sum_{m=0}^{l-1}(\sigma_{k+m+l} - \sigma_{k-m})\right) - \exp\left(\sum_{m=0}^{l-1}(\sigma_{k+m} - \sigma_{k-l-m})\right)\right) + \dot{\sigma}_k \sum_{m=1}^{l-1}(\dot{\sigma}_{k+m} - \dot{\sigma}_{k-m}) \tag{2}$$

and

$$\ddot{\sigma}_k = C\left(\exp\left(\sum_{m=1}^{l}(\sigma_{k-m} - \sigma_{k+l+m})\right) - \exp\left(\sum_{m=1}^{l}(\sigma_{k-l-m} - \sigma_{k+m})\right)\right) - \dot{\sigma}_k \sum_{m=1}^{l}(\dot{\sigma}_{k+m} - \dot{\sigma}_{k-m}). \tag{3}$$

Here C is an arbitrary constant, which is assumed to be unequal to zero. (If $C = 0$, then equations (2) and (3) are the Bogoyavlenskii lattice [24].) Equations (2) with $l = 1$ are evidently reduced to the Toda lattice (1). The Belov–Chaltikian lattice [25] is equivalent to equations (3) if $l = 1$.

In the periodic case ($\sigma_{k+n} = \sigma_k, n$ is a period), the lattices admit additional reduction constraints

$$\sigma_{m+1+k} = -\sigma_{m+1-k} \qquad \text{if} \quad n = 2m + 1$$
$$\sigma_{m+k} = -\sigma_{m+1-k} \qquad \text{if} \quad n = 2m$$

or

$$\sigma_{m+k} = -\sigma_{m-k} \qquad \text{if} \quad n = 2m$$

($k = 0, \ldots, m$). A connection of these constraints in the Toda lattice case with the root systems of semisimple Lie algebras was established in [26]. Reductions

$$\sigma_{-k} = -\sigma_k$$

or

$$\sigma_{1-k} = -\sigma_k$$

are consistent with the lattice equations in the infinite dimensional case.

It is well known that the Boussinesque equation

$$v_{\tau\tau} = (a^2 v + c_2 v_{xx} + c_3 v^2)_{xx} \tag{4}$$

can be obtained as a result of the limiting procedure in the Toda lattice [27]. This procedure is suitable in equations (2) and (3) for arbitrary l and gives the same equation. Indeed, assuming

$$\sigma_k = \varepsilon u(\tau, x) \qquad \tau = \varepsilon^2 t \qquad x = \varepsilon k$$

and expanding $u(\tau, x + \varepsilon m)$ in the Taylor series, we have from equations (2) and (3)

$$u_{\tau\tau} = \frac{c_1}{\varepsilon^2} u_{xx} + c_2 u_{xxxx} + 2c_3 u_x u_{xx} + O(\varepsilon).$$

Equation (4) follows this equality after differentiation if we put

$$v = u_x + \frac{c_1 - \varepsilon^2 a^2}{2c_3 \varepsilon^2}$$

and consider limit $\varepsilon \to 0$.

Equations (2) and (3) with arbitrary l were revealed in [11] to belong to a class of nonlinear equations admitting a representation as the compatibility condition of overdetermined linear systems (Lax pairs), whose coefficients are connected in an explicit manner. Thus, the Lax pair for equations (2) has the form

$$\begin{cases} -\dot{\psi}_k = \lambda \psi_{k-l} + \displaystyle\sum_{m=0}^{l-1} h_{k+m} \psi_k \\ z\psi_k = \lambda^2 \psi_{k-1} + \lambda h_{k+l-1} \psi_{k+l-1} + \rho_{k+2l-1} \psi_{k+2l-1} \end{cases} \tag{5}$$

where z and λ are complex parameters and

$$h_k = \dot{\sigma}_k$$

$$\rho_k = C \exp\left(\sum_{m=0}^{l-1} (\sigma_{k+m} - \sigma_{k-l-m})\right).$$

Also, this lattice is the compatibility condition of the so-called dual Lax pair

$$\begin{cases} \dot{\xi}_k = \lambda \xi_{k+l} + \displaystyle\sum_{m=0}^{l-1} h_{k+m} \xi_k \\ z\xi_k = \lambda^2 \xi_{k+1} + \lambda h_k \xi_{k-l+1} + \rho_k \xi_{k-2l+1}. \end{cases} \tag{6}$$

Direct and dual Lax pairs of equations (3) read respectively

$$\begin{cases} -\dot{\psi}_k = \lambda \psi_{k-l} - \displaystyle\sum_{m=1}^{l} h_{k+m} \psi_k \\ z\psi_k = \lambda^2 \psi_{k+1} + \lambda h_{k+l+1} \psi_{k+l+1} + \rho_{k+2l+1} \psi_{k+2l+1} \end{cases} \tag{7}$$

and

$$\begin{cases} \dot{\xi}_k = \lambda \xi_{k+l} - \displaystyle\sum_{m=1}^{l} h_{k+m} \xi_k \\ z\xi_k = \lambda^2 \xi_{k-1} + \lambda h_k \xi_{k-l-1} + \rho_k \xi_{k-2l-1}. \end{cases} \tag{8}$$

Here

$$\rho_k = C \exp\left(\sum_{m=1}^{l} (\sigma_{k-l-m} - \sigma_{k+m})\right).$$

It is worth noting that the dependence on λ of the Lax pairs (5) and (8) is the same as in the Lax pairs for the Nahm equations. Consequently, lattices (2) and (3) can be obtained from the Nahm equations by imposing appropriate reduction constraints.

We are able to rewrite the second equations of the Lax pairs (5) and (7) as infinite dimensional spectral problems

$$z\psi = L_1\psi$$

and

$$z\psi = L_2\psi$$

with the help of matrices L_1 and L_2 such that

$$\begin{aligned} L_{1,kj} &= \lambda^2\delta_{k,j+1} + \lambda h_j\delta_{k,j+1-l} + \rho_j\delta_{k,j+1-2l} \\ L_{2,kj} &= \lambda^2\delta_{k,j-1} + \lambda h_j\delta_{k,j-1-l} + \rho_j\delta_{k,j-1-2l}. \end{aligned}$$

The quantities

$$I_n = \operatorname{Tr} L_1^{nl}$$

and

$$J_n = \operatorname{Tr} L_2^{-nl}$$

($n \in \mathbb{N}$) give respectively the infinite hierarchies of the integrals of lattices (2) and (3). The first nontrivial integrals are

$$\begin{aligned} I_2 &= l\sum_{k=-\infty}^{\infty}\left(2\rho_k + h_k^2 + 2h_k\sum_{m=1}^{l-1}h_{k+m}\right) \\ J_2 &= l\sum_{k=-\infty}^{\infty}\left(-2\rho_k + h_k^2 + 2h_k\sum_{m=1}^{l}h_{k+m}\right). \end{aligned}$$

It is seen that a positively defined integral exists only in the case of real-valued solutions of the Toda lattice with $C > 0$.

At the end of this section, we present formulae for the expansions in series of the Lax pair solutions. One can put $\lambda = 1$ in the infinite dimensional case without loss of generality. Then the solutions of Lax pairs (5) and (6) admit in the neighbourhood of the point $z = \infty$ the following asymptotic expansions:

$$\psi_k = \alpha^k\left(1 + \frac{a_k}{z^l} + \frac{b_k}{z^{2l}} + \cdots\right)\mathrm{e}^{-t/\alpha^l} \tag{9}$$

$$\xi_k = \alpha^{-k}\left(1 - \frac{a_{k-l}}{z^l} + \frac{c_k}{z^{2l}} + \cdots\right)\mathrm{e}^{t/\alpha^l} \tag{10}$$

where

$$a_k = \sum_{m=-\infty}^{k+l-1} h_m$$

$$b_k = \sum_{m=-\infty}^{k+2l-1}(\rho_m - C + h_{m-l}a_{m-l}) \qquad c_k = \sum_{m=-\infty}^{k-1}(C - \rho_m + h_m a_{m-2l+1})$$

and

$$\alpha = \frac{1}{z}\left(1 + \frac{C}{z^{2l}} + \cdots\right)$$

is the solution of the equation

$$z\alpha = 1 + C\alpha^{2l}.$$

The solutions of Lax pairs (7) and (8) are represented in the next form

$$\psi_k = \beta^k(1 + z^l d_k + z^{2l} e_k + \cdots)\,\mathrm{e}^{-t/\beta^l} \tag{11}$$

$$\xi_k = \beta^{-k}(1 - z^l d_{k-l} + z^{2l} f_k + \cdots)\,\mathrm{e}^{t/\beta^l} \tag{12}$$

in the neighbourhood of the point $z = 0$. Here

$$d_k = -\sum_{m=-\infty}^{k+l} h_m$$

$$e_k = \sum_{m=-\infty}^{k+2l}(C - \rho_m - h_{m-l}d_{m-l}) \qquad f_k = \sum_{m=-\infty}^{k}(\rho_m - C - h_m d_{m-2l-1})$$

and

$$\beta = z(1 - z^{2l}C + \cdots)$$

satisfy equation

$$z = \beta + C\beta^{2l+1}.$$

These formulae for the expansions of the solutions are checked by immediate substitution into the Lax pairs.

3. Darboux transformation technique

In this section, we give the formulae of the Darboux transformations (DTs), which allow us to generate the infinite hierarchies of the solutions of lattices (2) and (3) together with those of their Lax pairs. The infinitesimal symmetries of the lattices are also found.

Let φ_k be a solution of system (5) with $z = x$ and $\lambda = \mu$. Equation (2) and its Lax pairs (5) and (6) are covariant with respect to DT

$$\tilde{\psi}_k = \dot{\psi}_k - \frac{\dot{\varphi}_k}{\varphi_k}\psi_k \tag{13}$$

$$\tilde{\xi}_k = \frac{\Delta_k}{\varphi_{k-l}} \tag{14}$$

$$\tilde{\sigma}_k = \sigma_k + \log\frac{\varphi_{k-l+1}}{\varphi_{k-l}} \tag{15}$$

where

$$\Delta_k = \int_{t_0}^{t} \varphi_{k-l}\xi_k\,\mathrm{d}t + \delta_k \tag{16}$$

constants δ_k are determined by equations

$$\lambda\delta_{k+l} - \mu\delta_k = \varphi_k\xi_k|_{t=t_0}$$

$$x\lambda^2\delta_{k+1} - z\mu^2\delta_k = [\lambda\mu h_k\varphi_k\xi_{k-l+1} + \lambda\rho_{k+l}\varphi_{k+l}\xi_{k-l+1} + \mu\rho_k\varphi_k\xi_{k-2l+1}]|_{t=t_0}.$$

The statement remains valid, when quantities Δ_k are defined as

$$\Delta_k = \frac{1}{\mu} \sum_{m=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^m \varphi_{k-ml} \xi_{k-ml} \tag{17}$$

or

$$\Delta_k = -\frac{1}{\mu} \sum_{m=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^m \varphi_{k+ml} \xi_{k+ml}. \tag{18}$$

We will refer to formulae (13)–(15) as the DT of the direct problem since the expressions for the transformed quantities depend explicitly on solution φ_k of direct Lax pair (5). If we carry out N iterations of transformation (13)–(15) on solutions $\varphi_k^{(j)}$ $(j = 1, \ldots, N)$ of Lax pair (5) with $z = x^{(j)}$ and $\lambda = \mu^{(j)}$, then the new (transformed) solution of equation (2) is

$$\tilde{\sigma}_k = \sigma_k + \log \frac{W\left(\varphi_{k-l+1}^{(1)}, \ldots, \varphi_{k-l+1}^{(N)}\right)}{W\left(\varphi_{k-l}^{(1)}, \ldots, \varphi_{k-l}^{(N)}\right)}. \tag{19}$$

Here we use notation $W\left(\varphi_k^{(1)}, \ldots, \varphi_k^{(N)}\right)$ for the Wronskian of functions $\varphi_k^{(1)}, \ldots, \varphi_k^{(N)}$.

In a similar manner, one can introduce the DT of the dual problem with the help of a solution of the dual Lax pair (6). The sum of the DTs of direct and dual problems is called the binary DT and yields the following expression for a new solution of lattice (2):

$$\tilde{\sigma}_k = \sigma_k + \log \frac{\Delta_{k+1}}{\Delta_k}. \tag{20}$$

Supposing $\lambda = \mu = 1$ for simplicity and considering limit $x = z + \delta z \to z$ under condition $\varphi_k = \psi_k + O(\delta z)$ in (20) we obtain

$$\tilde{\sigma}_k = \sigma_k + \delta z \delta\sigma_k + o(\delta z)$$

where

$$\delta\sigma_k = \Delta_{k+1} - \Delta_k$$

satisfy the linearization of equation (2). The substitution of expansions (9) and (10) into the last formula leads to the next representation

$$\delta\sigma_k = \sum_{m=1}^{\infty} \frac{\delta\sigma_k^{(m)}}{z^{ml}}$$

where quantities $\delta\sigma_k^{(m)}$ form the infinite hierarchy of the infinitesimal symmetries of lattice (2). The first members of the hierarchy are

$$\delta\sigma_k^{(1)} = h_k$$

$$\delta\sigma_k^{(2)} = \rho_{k+l} + \rho_k + h_k \sum_{m=-l+1}^{l-1} h_{k+m}.$$

Let us consider the zero background ($\sigma_k = 0$). The Lax pair solutions entering (19) have the form

$$\varphi_k^{(j)} = \sum_{m=1}^{2l} C_m^{(j)} \alpha_m^{(j)^k} \exp\left(-\mu^{(j)} \alpha_m^{(j)^{-l}} t\right) \tag{21}$$

where $\alpha_m^{(j)}$ $(m = 1, \ldots, 2l)$ are the roots of equations

$$x^{(j)} \alpha^{(j)} = \mu^{(j)^2} + C\alpha^{(j)^{2l}} \tag{22}$$

$C_m^{(j)}$ are arbitrary constants. Substituting (21) into equation (19), we find the multi-soliton solution of equation (2). (Note that, in the Toda lattice case, relations (22) can be considered as the equations which define $x^{(j)}$ and $\mu^{(j)}$ through given $\alpha_1^{(j)}$ and $\alpha_2^{(j)}$.) If all $x^{(j)}$ are different, then the one-soliton component of the multi-soliton solution of lattice (2) is characterized by $2l$ parameters: $x^{(j)}$ and $2l-1$ constants from set $C_m^{(j)}$ ($m = 1, \ldots, 2l$). These parameters determine the internal degrees of freedom of solitons. In the general case, the shape of the one-soliton solution ($N = 1$) changes under the evolution. If we put

$$C_1^{(j)} C_2^{(j)} \neq 0 \qquad C_m^{(j)} = 0 \ (m > 2)$$

then an interaction of the one-soliton components of this subvariety of the multi-soliton solutions can lead to their shifts only. Indeed, assuming

$$\begin{array}{lll} C > 0 & x^{(j)} > 0 & \mu^{(j)} \in \mathbb{R} \\ \alpha_1^{(j)} < \alpha_2^{(j)} & v_j > v_k & \text{if} \quad j < k \end{array}$$

where

$$v_j = \mu^{(j)} \frac{\alpha_1^{(j)^{-l}} - \alpha_2^{(j)^{-l}}}{\log\left|\alpha_1^{(j)}\right| - \log\left|\alpha_2^{(j)}\right|}$$

is the velocity of the one-soliton component, we obtain that the jth one-soliton component of the multi-soliton solution suffers shift

$$\delta_{jk} = \frac{1}{\mu^{(j)}\left(\alpha_2^{(j)^{-l}} - \alpha_1^{(j)^{-l}}\right)} \log \frac{\left(\mu^{(j)}\alpha_1^{(j)^{-l}} - \mu^{(k)}\alpha_1^{(k)^{-l}}\right)\left(\mu^{(j)}\alpha_2^{(j)^{-l}} - \mu^{(k)}\alpha_2^{(k)^{-l}}\right)}{\left(\mu^{(j)}\alpha_2^{(j)^{-l}} - \mu^{(k)}\alpha_1^{(k)^{-l}}\right)\left(\mu^{(j)}\alpha_1^{(j)^{-l}} - \mu^{(k)}\alpha_2^{(k)^{-l}}\right)}$$

after the interaction with the kth component. If the multi-soliton solution is nonsingular for $t \to -\infty$ and some of δ_{jk} are complex, then the solution became singular after the interaction. The interaction of solitons of general form is more complicated and changes their internal degrees of freedom. Since $\mu^{(j)}$ can differ by the sign for fixed set of the equation (22) roots, equation (2), as the Toda lattice equation, has the solitons propagating on the lattice in both directions.

In the case of lattice (3), we have the following formulae of the DT of the direct problem:

$$\tilde{\psi}_k = \dot{\psi}_k - \frac{\dot{\varphi}_k}{\varphi_k}\psi_k \tag{23}$$

$$\tilde{\xi}_k = \frac{\Delta_k}{\varphi_{k-l}} \tag{24}$$

$$\tilde{\sigma}_k = \sigma_k + \log\frac{\varphi_{k-l-1}}{\varphi_{k-l}}. \tag{25}$$

Here ψ_k and ξ_k are solutions of Lax pairs (7) and (8), φ_k is the solution of (7) with $z = x$, $\lambda = \mu$, Δ_k are defined by equations (16), where constants δ_k have to satisfy equations

$$\lambda\delta_{k+l} - \mu\delta_k = \varphi_k\xi_k|_{t=t_0}$$

$$x\lambda^2\delta_{k-1} - z\mu^2\delta_k = [\lambda\mu h_k\varphi_k\xi_{k-l-1} + \lambda\rho_{k+l}\varphi_{k+l}\xi_{k-l-1} + \mu\rho_k\varphi_k\xi_{k-2l-1}]|_{t=t_0}.$$

(Relations (17) or (18) can also be used as the definitions of Δ_k.) Iterating this DT N-times, we obtain the following expression for the new solution of lattice (3):

$$\tilde{\sigma}_k = \sigma_k + \log\frac{W\left(\varphi_{k-l-1}^{(1)}, \ldots, \varphi_{k-l-1}^{(N)}\right)}{W\left(\varphi_{k-l}^{(1)}, \ldots, \varphi_{k-l}^{(N)}\right)} \tag{26}$$

where $\varphi_k^{(j)}$ ($j = 1, \ldots, N$) are solutions of Lax pair (7) with $z = x^{(j)}$, $\lambda = \mu^{(j)}$.

As for the previous case, one can introduce the DT of the dual problem and construct the binary DT, which gives the following formula for the transformed solution of lattice (3):

$$\tilde{\sigma}_k = \sigma_k + \log \frac{\Delta_{k-1}}{\Delta_k}. \tag{27}$$

Let us suppose $\lambda = \mu = 1$ and consider limit $x = z + \delta z \to z$ provided that $\varphi_k = \psi_k + O(\delta z)$. Then equations (27) yield

$$\tilde{\sigma}_k = \sigma_k + \delta z \delta\sigma_k + o(\delta z)$$

where

$$\delta\sigma_k = \Delta_{k-1} - \Delta_k$$

is the solution of the linearization of equations (3). After substitution of expansions (11) and (12) into this formula we can derive

$$\delta\sigma_k = \sum_{m=1}^{\infty} \delta\sigma_k^{(m)} z^{ml}.$$

The first members of the hierarchy of the infinitesimal symmetries $\delta\sigma_k^{(m)}$ of lattice (3) are

$$\delta\sigma_k^{(1)} = h_k$$

$$\delta\sigma_k^{(2)} = \rho_{k+l} + \rho_k - h_k \sum_{m=-l}^{l} h_{k+m}.$$

Solutions $\varphi_k^{(j)}$ of the Lax pair (7) on the zero background are represented in the following manner,

$$\varphi_k^{(j)} = \sum_{m=1}^{2l+1} D_m^{(j)} \beta_m^{(j)k} \exp\left(-\mu^{(j)} \beta_m^{(j)^{-l}} t\right) \tag{28}$$

where $\beta_m^{(j)}$ $(m = 1, \ldots, 2l+1)$ are the roots of equations

$$x^{(j)} = \mu^{(j)^2} \beta^{(j)} + C \beta^{(j)^{2l+1}} \tag{29}$$

$D_m^{(j)}$ are constants. Substitution of (28) into (26) gives the multi-soliton solution of lattice (3). The properties of this solution are similar to those of lattice (2). If all $x^{(j)}$ are different, then the one-soliton component of the multi-soliton solution is completely described by parameter $y^{(j)}$ and $2l$ constants from set $D_m^{(j)}$ $(m = 1, \ldots, 2l+1)$. If only two constants $D_m^{(j)}$ for any j are unequal to zero, then an interaction of solitons can lead to their shifts. The expressions for the shifts differ from those for equations (2) by the notation (compare equations (21) and (28)).

It was shown in [12] that there are two families of the multi-soliton solutions of the nonlocal two-dimensional Toda lattice, which are produced by the DTs of direct and dual problems, respectively. These families differ since the dimension of the spaces of solutions of the corresponding Lax pair is infinite. In the case of lattices (2) and (3), the spaces of solutions of the Lax pairs have finite dimension, and, therefore, the multi-soliton solutions that are obtained by means of the DT of the direct problem and the DT of the dual problem coincide.

4. Conclusion

We have obtained expressions for the first integrals, infinitesimal symmetries and the multi-soliton solutions for the sets (2) and (3) of the lattice equations of the Toda type. These equations generalize the Toda lattice in the case of the systems of particles interacting with a few neighbourhoods and can be considered as the reductions of the Nahm equations. The evolution of the solutions of the lattices can result in the appearance of new singularities. These singularities do not arise in the case of real-valued solutions of the Toda lattice only, when a positively defined first integral exists. We have also found the expansions in series of the Lax pairs solutions and shown that the continuous limit of the lattices is the Boussinesque equation as in the Toda lattice case.

The equations considered in this paper do not fit into the class of lattices described in [7, 8]. The study of compatible flows, symmetries and discretizations of the lattice equations [8, 28, 29] can lead to new hierarchies of the integrable equations. From the point of view of the quantization of the equations considered, it is important to find the hierarchies of their Poisson structures and to include them into the r-matrix approach [30]. This is also significant for proving the complete integrability of the lattices in the periodic and infinite dimensional cases.

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